Regular Matroids with Graphic Cocircuits

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We introduce the notion of graphic cocircuits and show that a large class of regular matroids with graphic cocircuits belongs to the class of signed-graphic matroids. Moreover, we provide an algorithm which determines whether a cographic matroid with graphic cocircuits is signed-graphic or not.

1 Introduction

In this paper we examine the effect of removing cocircuits from regular matroids and we focus on the case in which such a removal always results in a graphic matroid. The first main result, given in section 3, is that a regular matroid with graphic cocircuits is signed-graphic if and only if it does not contain two specific minors. This provides a useful connection between graphic, regular and signed-graphic matroids which may be further utilised for devising combinatorial recognition algorithms for certain classes of matroids. At this point we should note that decomposition theories and recognition algorithms for matroids have provided some of the most important results of matroid theory and combinatorial optimization (see e.g. decomposition of graphic matroids [16] and recognition of network matrices [2] and decomposition of regular matroids [10] and recognition of totally unimodular matrices [11]). Finally, in section 4 we provide a simple recognition algorithm which determines whether a cographic matroid with graphic cocircuits is signed-graphic or not.

2 Preliminaries

In this section we will mention all the necessary definitions and preliminary results regarding graphs, signed graphs and their corresponding matroids. The definitions for graphs presented in this section are taken from [4, 19], while for signed graphs from [20, 21]. The main reference for matroid theory is the book of Oxley [8] while the main reference for signed-graphic matroids is [22].

2.1 Graphs

A graph $G := (V,E)$ is defined as a finite set of vertices V, and a set of edges $E \subseteq V \cup V^2$ where identical elements are allowed. Therefore we will have four types of edges: $e = \{u,v\}$ is called a *link*, $e = \{v,v\}$ a *loop*, $e = \{v\}$ a *half edge*, while $e = \emptyset$ is a *loose edge*. Whenever applicable, the vertices that define an edge are called its *end-vertices*. We say that a vertex v of a graph G is *incident with* an edge e of G and that e is *incident with* v if $v \in e$. We also say that two vertices u and v of G are *adjacent* or that u is *adjacent to* v if $\{u,v\}$ is an edge of G. Observe that the above is the ordinary definition of a graph, except that we also allow half edges and loose edges. We will denote the set of vertices and the set of edges of a graph G by $V(G)$ and $E(G)$, respectively.

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In what follows we will assume that we have a graph G. The following operations are defined. We say that G' is a *subgraph* of G, denoted by $G' \subseteq G$, if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. For some $X \subseteq V(G)$ the subgraph *induced* by X is defined as $G[X] := (X, E')$, where $E' \subseteq E(G)$ is a maximal set of edges with all end-vertices in X. If $X \subseteq E(G)$ then $G[X] := (V', X)$, where $V' \subseteq V(G)$ is the set of end-vertices of edges in X. The *deletion of an edge e* from G is the subgraph defined as $G \backslash e := (V(G), E - e)$. *Identifying* two vertices u and v is the operation where we replace u and v with a new vertex v' in both $V(G)$ and $E(G)$. The *contraction of a link* $e = \{u, v\}$ is the subgraph, denoted by G/e, which results from G by identifying u, v in $G \backslash e$. The *contraction of a half edge* $e = \{v\}$ or a *loop* $e = \{v, v\}$ is the subgraph, denoted by G/e, which results from the removal of $\{v\}$ and all half edges and loops incident with it, while all links incident with v become half edges at their other end-vertex. Contraction of a loose edge is the same as deletion. The *deletion of a vertex v* of G is defined as the deletion of all edges incident with v and the deletion of v from $V(G)$. A graph G' is called a *minor* of G if it is obtained from a sequence of deletions and contractions of edges and deletions of vertices of G.

Two graphs G and H are called *isomorphic*, and we write $G \cong H$, if there exists a bijection $p : V(G) \to V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{p(u), p(v)\} \in E(H)$. A *walk* in G is a sequence $(v_1, e_1, v_2, e_2, \dots, e_{t-1}, v_t)$ where e_i is incident with both v_i and v_{i+1}. If $v_1 = v_t$, then we say that the walk is *closed*. If a walk has distinct inner vertices, then it is called a *path*. The subgraph of G induced by the edges of a closed path is called a *cycle*. The edge set of a cycle of G is called a *circle* of G. A graph is called a *wheel graph*, if it consists of a cycle along with a vertex which is adjacent to every vertex of the cycle.

A graph is *connected* if there is a walk between any pair of its vertices. There are several notions of higher connectivity in graphs that have appeared in the literature. Here we will define *Tutte k-connectivity* which we shall call simply *k-connectivity*. For $k \geq 1$, a *k-separation* of a connected graph G is a partition (A, B) of the edges such that $\min\{|A|, |B|\} \geq k$ and $|V(G[A]) \cap V(G[B])| = k$. For $k \geq 2$, we say that G is *k-connected* if G does not have an l-separation for $l = 1, \dots, k-1$. Note that our notion of k-connectivity of a graph is taken from Tutte's graph theory book [19] which is different from the notion of k-connectivity we find in other graph theory books, e.g. [3, 4]. We use k-connectivity as defined above in this paper, due to the fact that the connectivity of a graph and its corresponding graphic matroid coincide under this definition.

2.2 Signed graphs

A *signed graph* is defined as $\Sigma := (G, \sigma)$ where G is a graph called the *underlying graph* and σ is a sign function $\sigma : E(G) \to \{\pm 1\}$, where $\sigma(e) = -1$ if e is a half edge and $\sigma(e) = +1$ if e is a loose edge. Therefore a signed graph is a graph where the edges are labelled as positive or negative, while all the half edges are negative and all the loose edges are positive. We denote by $V(\Sigma)$ and $E(\Sigma)$ the vertex set and edge set of a signed graph Σ, respectively.

The *sign of a cycle* is the product of the signs of its edges, so we have a *positive cycle* if the number of negative edges in the cycle is even, otherwise the cycle is a *negative cycle*. Both negative loops and half-edges are negative cycles. A signed graph is called *balanced* if it contains no negative cycles. Finally, although signed graphs have been studied extensively, it is out of the scope of this work to provide more notions, definitions and results regarding signed graphs. However, the interested reader is referred to [20, 22, 23]

2.3 Matroids

Definition 1. *A matroid M is an ordered pair $(E, \mathscr{I})$ of a finite set E and a collection $\mathscr{I}$ of subsets of E satisfying the following three conditions:*

(I1) $\emptyset \in \mathscr{I}$

(I2) If $X \in \mathscr{I}$ and $Y \subseteq X$ then $Y \in \mathscr{I}$

(I3) If U and V are members of $\mathscr{I}$ with $|U| < |V|$ then there exists $x \in V - U$ such that $U \cup x \in \mathscr{I}$.

Given a matroid $M = (E, \mathscr{I})$, the set E is called the *ground set* of M and the members of $\mathscr{I}$ are the *independent sets* of M; furthermore, any subset of E not in $\mathscr{I}$ is called a *dependent set* of M. A minimal dependent set is called a *circuit* of M. The *rank function* $r_M : 2^E \to \mathbb{Z}_+$ of a matroid M is a function defined by: $r_M(A) = max(|X| : X \subseteq A, X \in \mathscr{I})$, where $A \subseteq E$ and $|A|$ is the cardinality of A. The axiomatic Definition 1 for a matroid on a given ground set uses its independent sets. However, there are several equivalent ways to define a matroid which can be found in [8]. For example, a matroid M on a given ground set E can be defined through its rank function or through its set of circuits. We provide here the following axiomatisation of a matroid by its circuits [8]:

Proposition 2.1. *A collection $\mathscr{C}$ of subsets of E is the collection of circuits of a matroid on E if and only if $\mathscr{C}$ satisfies the following conditions:*

(I1) $\emptyset \notin \mathscr{C}$

(I2) If C_1 and C_2 are members of $\mathscr{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(I3) If C_1 and C_2 are distinct members of $\mathscr{C}$ and $e \in C_1 \cap C_2$, then there is a $C_3 \in \mathscr{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

Two matroids M_1 and M_2 are called *isomorphic* if there is a bijection ψ from $E(M_1)$ to $E(M_2)$ such that $X \in \mathscr{I}(M_1)$ if and only if $\psi(X) \in \mathscr{I}(M_2)$. We denote that M_1 and M_2 are isomorphic by $M_1 \cong M_2$.

Let E be a finite set of vectors from a vectorspace over some field F and let $\mathscr{I}$ be the collection of all subsets of linearly independent elements of E; then it can be proved that $M = (E, \mathscr{I})$ is a matroid called *vector matroid*. Furthermore, any matroid isomorphic to M is called a *representable matroid* over F. Matroids representable over the finite field $GF(2)$ are called *binary* and matroids representable over the finite field $GF(3)$ are called *ternary*. A matroid representable over every field is a *regular* matroid. Let A be a matrix whose columns are the vectors of the ground set of a vector matroid M. It is evident that there is one-to-one correspondence between the linearly independent columns of A and the independent sets of M, so the matroid M can be fully characterized by matrix A. Matrix A is called a *representation matrix* of M and we denote the vector matroid with representation matrix A by $M[A]$. Let G be a graph without loops, half-edges or loose edges and let $\mathscr{C}$ be the collection of edge sets of cycles of G. Then it can be shown that the pair $(E(G), \mathscr{C})$ is a matroid called the *cycle matroid* of G and is denoted by $M(G)$. A matroid M such that $M \cong M(G)$ is called *graphic*.

Given a matroid $M = (E, \mathscr{I})$, the ordered pair $(E, \{E - S : S \notin \mathscr{I}\})$ is a matroid called the *dual matroid* of M and denoted by M^*. There is always a dual matroid M^* associated with a matroid M and it is clear that $(M^*)^* = M$. Usually, the prefix 'co' is used to dualize a term. Therefore, the set $\mathscr{C}(M^*)$ of circuits of M^* is the set of *cocircuits* of M. We usually denote the cocircuit of M by $\mathscr{C}^*(M)$.

Deletion and *contraction* are two fundamental matroid operations. Formally, given a matroid $M = (E, \mathscr{C})$ on a ground set E defined by its collection of circuits $\mathscr{C}$ the *deletion of* some $T \subseteq E$ from M is the matroid denoted by $M \backslash T$, on $E \backslash T$ with the following collection of circuits:

$$\mathscr{C}(M \backslash T) := \{C \in \mathscr{C}(M) | C \cap T = \emptyset\}. \tag{1}$$

The *contraction of* some $T \subseteq E$ is the matroid denoted by M/T, on $E\backslash T$ with the following collection of circuits:

$$\mathscr{C}(M/T) := \text{minimal}\{C\backslash T | C \in \mathscr{C}(M)\}. \tag{2}$$

Furthermore, deletion and contraction may be viewed as dual operations in the sense that the deletion or contraction of a set $T \subseteq E(M)$ from M is translated as the contraction or deletion of T from M^*, respectively. In a symbolic way this is expressed as follows:

$$M\backslash T = (M^*/T)^* \text{ and } M/T = (M^*\backslash T)^* \tag{3}$$

Any matroid which can be obtained from M by a series of deletions and contractions is called a *minor* of M. If M has a minor isomorphic to a matroid N then we will often say that M has an *N-minor* or M has N *as a minor*. A matroid N is called an *excluded minor* for a class of matroids $\mathscr{M}$ if $N \notin \mathscr{M}$ but every proper minor of N is in $\mathscr{M}$. A well-known excluded minor characterization for graphic matroids goes as follows [18], where K_5 is the complete graph on five vertices and $K_{3,3}$ is the complete bipartite graph having three vertices at each side of the bipartition:

Theorem 2.1. *A regular matroid is graphic if and only if it has no minor isomorphic to $M^*(K_{3,3})$ or $M^*(K_5)$.*

Consider a matroid M defined by a rank function $r : E(M) \to \mathbb{Z}$. For some positive integer k, a partition (X,Y) of $E(M)$ is called a *k-separation* of M if the following two conditions are satisfied:

(M1) $\min\{|X|,|Y|\} \geq k$, and

(M2) $r_M(X) + r_M(Y) - r(M) \leq k-1$.

If a matroid M is k-separated for some integer k then the *connectivity of a matroid* of M is the smallest integer j for which M is j-separated; otherwise, we take the connectivity of M to be infinite.

2.4 Signed-Graphic Matroids

The definition of the *signed-graphic matroid* goes as follows [20]:

Theorem 2.2. *Given a signed graph Σ let $\mathscr{C} \subseteq 2^{E(\Sigma)}$ be the family of minimal edge sets inducing a subgraph in Σ which is either:*

(a) a positive cycle, or

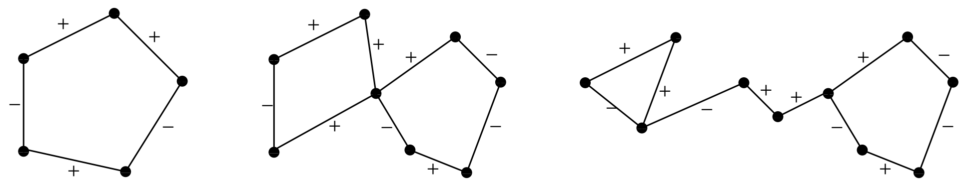
(b) two negative cycles which have exactly one common vertex, or

(c) two vertex-disjoint negative cycles connected by a path which has no common vertex with the cycles apart from its end nodes.

Then $M(\Sigma) = (E(\Sigma), \mathscr{C})$ is a matroid on $E(\Sigma)$ with circuit family $\mathscr{C}$.

The subgraphs of Σ induced by the edges corresponding to a circuit of $M(\Sigma)$ are called the *circuits* of Σ. Therefore a circuit of Σ can be one of three types (see Figure 2.4 for example circuits of types (a), (b) and (c)).

For each signed graph Σ with edge set $E(\Sigma)$, there is an associated signed-graphic matroid $M(\Sigma)$ on the set of elements $E(\Sigma)$. However for a given signed-graphic matroid M there may exist several signed graphs Σ_i such that $M = M(\Sigma_i)$ where $i \geq 1$. So signed-graphic matroids can be viewed as the abstract entities, while their corresponding signed graphs their representations in a graphical context.



(a) (b) (c)

Figure 1: Circuits in a signed graph Σ

3 An excluded minor characterization

We say that a cocircuit Y of a binary matroid M is *graphic* if $M\backslash Y$ is a graphic matroid; otherwise, we say that Y is a *non-graphic* cocircuit. Two important theorems which associate signed-graphic matroids with cographic matroids and regular matroids in terms of excluded minors have been shown by Slilaty et.al. in [9, 12]. Specifically, of the 35 forbidden minors for projective planar graphs 29 are not 1-separable; these 29 graphs, which we call $G_1, G_2, \ldots, G_{29}$, can be found in [1, 6]. Slilaty has shown in [12] that the collection of the cographic matroids of these 29 graphs $\{M^*(G_1), M^*(G_2), \ldots, M^*(G_{29})\}$, forms the complete list of the cographic excluded minors for signed-graphic matroids. Since cographic matroids is a subclass of regular matroids (see [8]), we expect the list of regular excluded minors for signed-graphic matroids to contain the matroids in $\mathscr{M}$ and some other matroids. Those other matroids are the R_{15} and R_{16} whose representation matrices over $GF(2)$ are the following:

$$\underbrace{\begin{bmatrix} 1&0&0&0&0&0&0&1&0&1&0&0&0&0&1\\ 0&1&0&0&0&0&0&0&0&0&1&1&0&1&0\\ 0&0&1&0&0&0&0&1&1&0&0&1&1&0&0\\ 0&0&0&1&0&0&0&1&1&0&0&0&1&1&0\\ 0&0&0&0&1&0&0&0&1&1&1&1&0&0&0\\ 0&0&0&0&0&1&0&0&1&1&1&1&1&0&0\\ 0&0&0&0&0&0&1&0&1&1&1&1&1&0&1 \end{bmatrix}}_{R_{15}}, \underbrace{\begin{bmatrix} 1&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&1&1&1&0\\ 0&0&1&0&0&0&0&0&0&1&1&0&1&1&1&0\\ 0&0&0&1&0&0&0&0&0&0&0&1&1&1&0&0\\ 0&0&0&0&1&0&0&0&1&1&0&1&1&1&0&0\\ 0&0&0&0&0&1&0&0&1&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&1 \end{bmatrix}}_{R_{16}}.$$

Specifically, in [12] we find Theorem 3.1 and in [9] we find Theorem 3.2.

Theorem 3.1. *A cographic matroid M is signed-graphic if and only if M has no minor isomorphic to $M^*(G_1), \ldots, M^*(G_{29})$.*

Theorem 3.2. *A regular matroid M is signed-graphic if and only if M has no minor isomorphic to $M^*(G_1), \ldots, M^*(G_{29}), R_{15}$ or R_{16}.*

The following two lemmas are essential for the proof of the main result of this section which characterizes the regular matroids with graphic cocircuits.

Lemma 3.3. *If a matroid M is isomorphic to $M^*(G_{17})$ or $M^*(G_{19})$ then for any cocircuit $Y \in \mathscr{C}^*(M)$, the matroid $M\backslash Y$ is graphic.*

Proof: By (3), we can equivalently show that, for any circuit Y of M^*, the matroid M^*/Y is cographic. The matroid M^* is graphic and thus, regular. Therefore, by Theorem 2.1, we have to show that for any circuit Y of $M^* \in \{M(G_{17}), M(G_{19}))\}$ the matroid M^*/Y has no minor isomorphic to $M(K_5)$ or $M(K_{3,3})$. Observe that G_{17} is isomorphic to the graph $K_{3,5}$ and G_{19} is isomorphic to $K_{4,4}\backslash e$, where e is any edge of $K_{4,4}$. Since $M(G_{19})$ is a graphic matroid we have that $M(G_{19}) \cong M(K_{4,4}\backslash e) = M(K_{4,4})\backslash e$.

Therefore, $M(K_{4,4})$ has a minor isomorphic to $M(G_{19})$ and by (1), any circuit of $M(G_{19})$ is a circuit of $M(K_{4,4})$. Thus, it suffices to prove that for any circuit $Y_1 \in \mathscr{C}(M(K_{3,5}))$ and $Y_2 \in \mathscr{C}(M(K_{4,4}))$ the matroids $M(K_{3,5})/Y_1 = M(K_{3,5}/Y_1)$ and $M(K_{4,4})/Y_2 = M(K_{4,4}/Y_2)$ have no minor isomorphic to $M(K_5)$ or $M(K_{3,3})$ in order to prove the theorem.

Since $K_{3,5}$ and $K_{4,4}$ are 3-connected, we get that Y_1 and Y_2 correspond to circles of $K_{3,5}$ and $K_{4,4}$, respectively. The 3-connected graphs $K_{3,5}$ and $K_{4,4}$ are also bipartite and therefore, they have no circle of odd cardinality and moreover, they have no parallel edges. This means that $K_{3,5}/Y_1$ and $K_{4,4}/Y_2$ have at most five vertices each. Therefore, the matroids $M(K_{3,5}/Y_1)$ and $M(K_{4,4}/Y_2)$ have rank at most 4 which is less than the rank of $M(K_{3,3})$. Evidently, $M(K_{3,5}/Y_1)$ and $M(K_{4,4}/Y_2)$ can not have a minor isomorphic to $M(K_{3,3})$.

It remains to be shown that $M(K_{3,5}/Y_1)$ and $M(K_{4,4}/Y_2)$ have no minor isomorphic to $M(K_5)$. Let us suppose that Y_1 and Y_2 are circuits of cardinality four. Then, since $K_{3,5}$ and $K_{4,4}$ are 3-connected we have that (by Lemma 5.3.2 in [8]) Y_1 and Y_2 are circles of $K_{3,5}$ and $K_{4,4}$, respectively, with cardinality four. Observe now that for any Y_1 and Y_2, the graphs $K_{3,5}/Y_1$ and $K_{4,4}/Y_2$ are isomorphic to the graphs $\overline{G}$ and $\hat{G}$ of Figure 2, respectively. Furthermore, parallel edges of a graph correspond to parallel elements in the associated graphic matroid. Therefore, any simple minor of $M(\overline{G})$ or $M(\hat{G})$ has at most seven or eight elements, respectively. The matroid $M(K_5)$ is simple and has ten elements. Therefore, $M(K_5)$ can not be a minor of $M(\overline{G}) \cong M(K_{3,5}/Y_1)$ or $M(\hat{G}) \cong M(K_{4,4}/Y_2)$. For the remaining case, that is, if Y_1 or Y_2 has more than four elements, the proof is quite similar to the one we followed in order to prove that $M(K_{3,5}/Y_1)$ and $M(K_{4,4}/Y_2)$ have no minor isomorphic to $M(K_{3,3})$ and for that reason is ommited. ☐

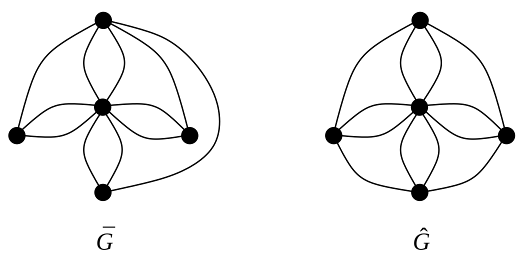


Figure 2: The graphs $\overline{G}$ and $\hat{G}$.

Lemma 3.4. *If N is a minor of a matroid M then for any cocircuit C_N of N there exists a cocircuit C_M of M such that $N\backslash C_N$ is a minor of $M\backslash C_M$.*

Proof: We have that $N = M\backslash X/Y$ for some disjoint $X,Y \subseteq E(M)$ and by duality, $N^* = M^*/X\backslash Y$. Therefore, by the definitions of deletion and contraction given in (1) and (2), we have that for any cocircuit $C_N \in \mathscr{C}(N^*)$ of N there exists a cocircuit $C_M \in \mathscr{C}(M^*)$ of M such that

(i) $C_N \subseteq C_M$,

(ii) $E(N) \cap C_M = C_N$,

which in turn imply that $C_M - C_N \subseteq X$. Therefore, $M\backslash X$ is a minor of $M\backslash\{C_M - C_N\}$ and since N is a minor of $M\backslash X$ we obtain that N is a minor of $M\backslash\{C_M - C_N\}$. By

$$M\backslash C_M = M\backslash\{C_M - C_N\}\backslash C_N$$

and the fact that N is a minor of $M\backslash\{C_M - C_N\}$ we have that $N\backslash C_N$ is a minor of $M\backslash C_M$. ☐

We are now ready to prove the main result of this section.

Theorem 3.5. *Let M be a regular matroid such that all its cocircuits are graphic. Then, M is signed-graphic if and only if M has no minor isomorphic to $M^*(G_{17})$ or $M^*(G_{19})$.*

Proof: The "only if" part is clear because of Theorem 3.1. For the "if" part, by way of contradiction, assume that M is not signed-graphic. By Theorem 3.1, M must contain a minor N which is isomorphic to some matroid in the set

$$\mathscr{M} = \{M^*(G_1),\dots,M^*(G_{16}),M^*(G_{18}),M^*(G_{20}),\dots,M^*(G_{29}),R^*_{15},R^*_{16}\}.$$

By case analysis, verified also by the MACEK software [5], it can be shown that for each matroid $M' \in \mathscr{M}$ there exists a cocircuit $Y' \in \mathscr{C}(M'^*)$ such that the matroid $M'\backslash Y'$ does contain an $M^*(K_{3,3})$ or an $M^*(K_5)$ as a minor. Therefore, by Theorem 2.1, there exists a cocircuit $Y_N \in \mathscr{C}(N^*)$ such that $N\backslash Y_N$ is not graphic. Therefore, by Lemma 3.4, there is a cocircuit $Y_M \in \mathscr{C}(M^*)$ such that $N\backslash Y_N$ is a minor of $M\backslash Y_M$. Thus, $M\backslash Y_M$ is not graphic which is in contradiction with our assumption that M has graphic cocircuits. $\square$

4 A recognition algorithm

Based on Theorem 3.5, we shall provide an algorithm which given a cographic matroid M with graphic cocircuits determines whether M is signed-graphic or not. In order to do this, we initially consider the following problem (P_0):
$(\mathbf{P_0})$: Find the members of the class $\mathscr{G}$ of 3-connected graphs defined as follows: $G \in \mathscr{G}$ if the cographic matroid $M^*(G)$ satisfies the following two conditions: (i) $M^*(G)$ has a minor isomorphic to $M^*(G_{17})$ or $M^*(G_{19})$, and (ii) for any $X \in \mathscr{C}^*(M(G))$, the matroid $M^*(G)\backslash X$ is graphic.

By duality, we obtain the following equivalent problem (P_1):
$(\mathbf{P_1})$: Find the members of the class $\mathscr{G}$ of 3-connected graphs defined as follows: $G \in \mathscr{G}$ if the graphic matroid $M(G)$ satisfies the following two conditions: (i) $M(G)$ has a minor isomorphic to $M(G_{17})$ or $M(G_{19})$, and (ii) for any $X \in \mathscr{C}(M(G))$, the matroid $M(G)/X$ is cographic.

Let M' be a matroid isomorphic to a minor of the graphic matroid $M(G)$. Since graphic matroids are closed under minors (see Corollary 3.2.2 in [8]), we have that there exists a graph G' such that $M(G') \cong M'$. This implies that there exist disjoint subsets S and T of $E(M(G))$ such that $M(G)\backslash S/T \cong M(G')$. By well-known results regarding the minors of graphic matroids (see results 3.1.2 and 3.2.1 in [8]), $M(G)\backslash S/T = M(G\backslash S/T) \cong M(G')$. By Lemma 5.3.2 in [8], if G' is 3-connected and $M(G\backslash S/T) \cong M(G')$ then $G' \cong \hat{G}$, where $\hat{G}$ is the graph obtained from $G\backslash S/T$ by deleting any isolated vertices. Thus, since both G_{17} and G_{19} are 3-connected, condition (i) of (P_1) is equivalent to: G has a $G_{17}-$ or a $G_{19}-$minor. By the dual version of Theorem 2.1 and due to the fact that K_5 and $K_{3,3}$ are 3-connected graphs, we have that condition (ii) of (P_1) is equivalent to: for any circle X of G, the graph G/X has no K_5- or $K_{3,3}-$minor. Furthermore, since there is one-to-one correspondence between the circles of a graph and the circuits of the associated graphic matroid, we easily obtain the following problem (P_2) which is equivalent to (P_1) and, therefore, equivalent to (P_0):

$(\mathbf{P_2})$: Find the members of the class $\mathscr{G}$ of 3-connected graphs defined as follows: $G \in \mathscr{G}$ if G satisfies the following two conditions: (i) G has a $G_{17}-$ or a $G_{19}-$minor, and (ii) for any circle X of G,

the graph G/X has no K_5- or $K_{3,3}-$minor.

In the following Theorem 4.2 we identify the members of $\mathscr{G}$ which have a $G_{17}-$minor and probably not a $G_{19}-$minor. Of great importance for the proof of this theorem is Theorem 4.1 of Negami in [7], which is a complement theorem to the well known Wheel Theorem of Tutte (see [17, 19]).

Theorem 4.1. *Let H be a graph not isomorphic to a wheel. Then a graph G is* 3*-connected and has H as a minor if and only if G can be obtained from H by a sequence of the following two operations:*

- *O1: addition of an edge between two non-adjacent vertices, and*
- *O2: the replacement of a vertex v of degree at least* 4 *by two adjacent vertices v_1 and v_2 such that each vertex formerly adjacent to v becomes adjacent to exactly one of v_1 or v_2 so that in the resulting graph the degree of each of v_1 and v_2 is greater than* 2*.*

Moreover, in Theorem 4.2 we denote by $K_{3,n}$ the complete bipartite graph with 3 and n vertices at each side of the bipartition and by $K^{+1}_{3,n}$, $K^{+2}_{3,n}$ and $K^{+3}_{3,n}$ are denoted the graphs which are isomorphic to the graphs depicted in Figure 3, where $n \geq 5$. Clearly, $K^{+1}_{3,n}$, $K^{+2}_{3,n}$ or $K^{+3}_{3,n}$ can be obtained from $K_{3,n}$ by a specific addition of one, two or three edges, respectively.

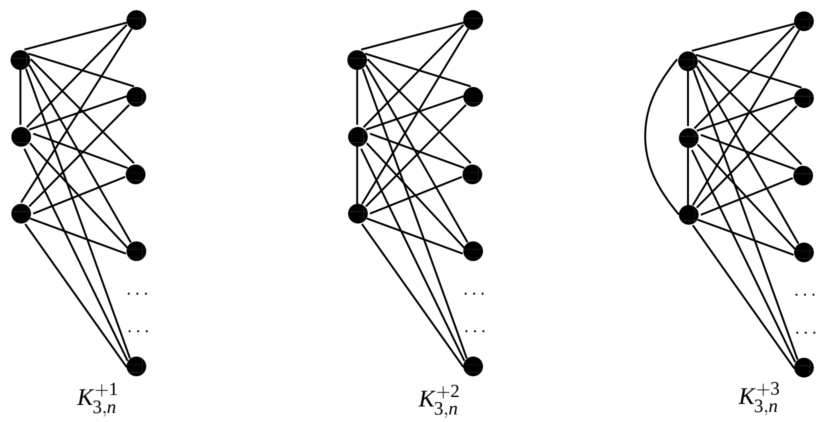


Figure 3: The graphs $K^{+1}_{3,n}$, $K^{+2}_{3,n}$ and $K^{+3}_{3,n}$.

Theorem 4.2. *A graph G is isomorphic to one of the graphs in $\mathscr{L} = \{K_{3,n}, K^{+1}_{3,n}, K^{+2}_{3,n}, K^{+3}_{3,n}\}$, where $n \geq 5$, if and only if G satisfies the following conditions:*

- *(i) it is* 3*-connected,*
- *(ii) it has a $G_{17}-$minor, and*
- *(iii) for any circle X of G, the graph G/X has neither a K_5 nor a $K_{3,3}$ as a minor.*

Proof: Let $\mathscr{F}$ be the class of graphs consisting of all the graphs satisfying conditions (i), (ii) and (iii) of the theorem. We shall show that $\mathscr{F} = \mathscr{L}$. Note that for any graph in $\mathscr{L}$ we denote by B the subset of its vertices having degree 3 and by A the set of the remaining vertices. Furthermore, if an O_2 operation is applied in a graph H in $\mathscr{L}$ such that a vertex v is replaced by two vertices v_1 and v_2 then in the graph G so obtained we say that A is the set of vertices of G obtained from the set A associated with H by deleting v and adding v_1 and v_2. We initially prove two claims.

Claim 1: Let G_e be a graph obtained from a graph $G \in \mathscr{L}$ by applying operation $O1$ and also let e be the

edge added by this operation. Then, if at least one end-vertex of e is in B then $G_e \notin \mathscr{F}$.
Proof of Claim 1: If one of the end-vertices of e is in A and the other is in B then e will have the same end-vertices with an existing edge of G and thus, the graph G_e so-obtained will not be 3-connected. For the remaining case, up to isomorphism, G_e will have as a subgraph the graph depicted in Figure 4. Contracting the circle consisting of the dashed edges we obtain a graph having a $K_{3,3}-$minor. Thus, in both cases we have that $G_e \notin \mathscr{F}$. ■

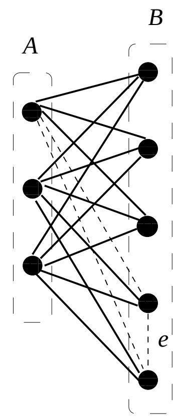


Figure 4: Operation $O1$ applied to $K_{3,5}$.

Claim 2: Let G_v be a graph obtained from a graph $G \in \mathscr{L}$ by applying operation $O2$ and also let v_1 and v_2 be the vertices of G_v replacing a vertex v of G due to this operation. If each of v_1 and v_2 is adjacent to at least one vertex of A and at least one vertex of B then $G_e \notin \mathscr{F}$.
Proof of Claim 2: Clearly v has to be a vertex of A in G since the vertices of B have degree 3 and thus, Operation $O2$ can not be applied. If we apply Operation $O2$ to G such that each of the vertices v_1 and v_2 so-created is adjacent to at least one vertex of A and at least one vertex of B then it is not difficult to check that the graph G_v so-obtained will have a subgraph isomorphic to one of the two graphs depicted in Figure 5. It is easy to see that the graph of Figure 5(i) contains a $K_{3,3}-$minor. In the remaining case, if G_v has a subgraph isomorphic to the graph depicted in Figure 5(ii) then if we contract the circle consisting of the dashed edges in this graph, we obtain a graph having $K_{3,3}$ as a minor and therefore, $G_v \notin \mathscr{F}$. ■

Suppose now that we start from $G_{17} \cong K_{3,5}$ and apply Theorem 4.1 in order to create the class of 3-connected graphs having $K_{3,5}$ as a minor. Clearly, operations $O1$ and $O2$ have to be applied. Due to Claim 1, in order to produce a graph which may be in $\mathscr{F}$, operation $O1$ must be applied such that the new edge added joins two vertices in the vertex set A of $K_{3,5}$; otherwise, a graph not in $\mathscr{F}$ is created. Because of Claim 2, in order to produce a graph which may be in $\mathscr{F}$, operation $O2$ can take place only after we have obtained a graph being $K_{3,5}^{+2}$ or $K_{3,5}^{+3}$ by a sequence of $O1$ operations; furthermore, the vertex v of $K_{3,5}^{+2}$ or $K_{3,5}^{+3}$ which is replaced by operation $O2$ has to be a vertex of A being adjacent to the other two vertices of A. Applying now operation $O2$ on $K_{3,5}^{+2}$ or $K_{3,5}^{+3}$ as described above we obtain $K_{3,6}$ or $K_{3,6}^{+1}$, respectively, both of which belong to $\mathscr{F}$. Similarly we can apply $O1$ and $O2$ on $K_{3,6}$ or $K_{3,6}^{+1}$ in the way implied by Claims 1 and 2 in order to create 3-connected graphs which may belong to $\mathscr{F}$. Continuing this process we get that all the possible 3-connected graphs so-obtained constitute a class which is equal to $\mathscr{L}$ and includes $\mathscr{F}$.

It remains to be shown that any member of $\mathscr{L}$ is a member of $\mathscr{F}$. Clearly any graph in $\mathscr{L}$ is 3-connected and has G_{17} as a minor. We shall show that there is no circle Y of a graph $H \in \mathscr{L}$ such that

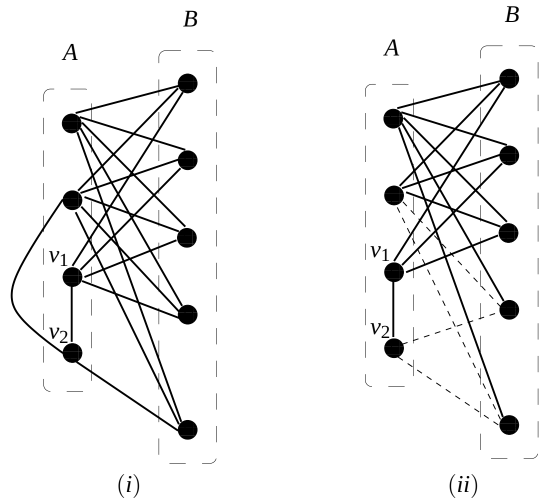


Figure 5: Operation $O2$ applied to $K_{3,5}^{+2}$.

H/X contains K_5 or $K_{3,3}$ as a minor. We firstly show that H does not contain K_5 as a minor which clearly implies that H/X has no K_5-minor. Observe that H has three vertices of degree more than 3 and that we would like to produce by a sequence of edge and vertex deletions and edge contractions the graph K_5 which has five vertices of degree 4. Clearly, the deletion of vertices or edges from a graph can not increase the degree of a vertex in the graph so-obtained; on the other hand, we can not obtain a graph from H by contracting edges which will have more than three vertices with degree greater than 3. We now show that H/Y does not have $K_{3,3}$ as a minor. Observe that any cycle of H contains at least two vertices belonging to the vertex set A of H. Thus, in H/Y there are at most two vertices with degree greater than 3. Evidently, no sequence of deletions and contractions of edges and deletions of vertices produces a graph which is isomorphic to $K_{3,3}$. □

In the following Theorem 4.3 we identify the members of $\mathscr{G}$ which have a $G_{19}-$minor and probably not a $G_{17}-$minor. As we did in Theorem 4.2 regarding the G_{17} case, we use Theorem 4.1 in order to identify these members.

Theorem 4.3. *A graph G is isomorphic to one of the graphs in $\mathscr{M} = \{K_{4,4}^-, K_{4,4}\}$ if and only if G satisfies the following conditions:*

(i) it is 3*-connected,*

(ii) it has a $G_{19}-$minor, and

(iii) for any circle X of G, the graph G/X has neither a K_5 nor a $K_{3,3}$ as a minor.

Proof: Let $\mathscr{H}$ be the class of graphs containing all the graphs satisfying conditions (i), (ii) and (iii) of the theorem. We shall show that $\mathscr{H} = \mathscr{M}$. Any graph in $\mathscr{L}$ is bipartite and thus, for a graph $G \in \mathscr{L}$ there exists a bipartition of $V(G)$ into two sets, which we denote by V_1 and V_2, such that no two adjacent vertices of G belong to the same set of the bipartition. We initially prove two claims.
Claim 1: Let G_e be a graph obtained from a graph $G \in \mathscr{M}$ by applying operation $O1$, where let e be the edge added by this operation. Then, if both end-vertices of e belong to either V_1 or V_2 then $G_e \notin \mathscr{H}$.
Proof of Claim 1: We prove the claim only for the case in which G is isomorphic to $K_{4,4}^-$, since the case in

which G is isomorphic to $K_{4,4}$ follows easily. There are two non-isomorphic graphs obtained by applying operation $O1$ such that e has both of its end-vertices in V_1 or in V_2; these graphs are depicted in Figure 6. In these graphs, if we contract the circle consisting of the dashed edges then we obtain a graph containing a minor isomorphic to $K_{3,3}$ and thus, the result follows. ■

Figure 6: Operation $O1$ applied to $K^-_{4,4}$.

Claim 2: If G_v is a graph obtained from a graph $G \in \mathscr{M}$ by applying operation $O2$ then $G_v \notin \mathscr{H}$.
Proof of Claim 2: If G is $K^-_{4,4}$ then any possible application of operation $O2$ to this graph would produce a graph being isomorphic to that depicted in (i) of Figure 7. Similarly, if G is $K_{4,4}$ then any possible application of operation $O2$ to this graph would produce a graph being isomorphic to that depicted in (ii) of Figure 7. In these graphs, if we contract the cycle consisting of the dashed edges then, in each case, we obtain a graph which has $K_{3,3}$ as a minor and thus, the result follows. ■

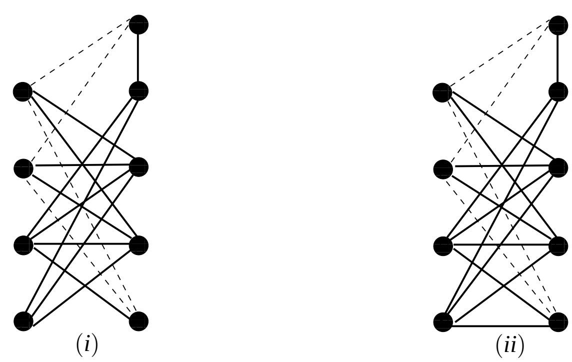


Figure 7: Operation 02 applied to $K^-_{4,4}$ and $K_{4,4}$.

In order to produce all the 3-connected graphs having $G_{19} \cong K^-_{4,4}$ as a minor Theorem 4.1 can be applied. However, due to Claim 2, if we apply operation $O2$ on $K^-_{4,4}$ or $K_{4,4}$ then the graph so-obtained is not in $\mathscr{H}$. Moreover, due to Claim 1, Operation O_1 may only add an edge joining vertices between different vertices of the vertex bipartition of $K^-_{4,4}$ in order to obtain a graph in $\mathscr{H}$. Thus, it is clear that the class of graphs satisfying conditions (i), (ii) and (iii) of the Theorem is a subclass of $\mathscr{M}$. We shall now complete the proof by showing that $K^-_{4,4}$ and $K_{4,4}$ are in $\mathscr{H}$. Clearly both are 3-connected and contain $G_{19} \cong K^-_{4,4}$ as a minor. Furthermore, any cycle in both graphs consists of at least four vertices and therefore, the contraction of the associated circle gives rise to a graph with at most five vertices which clearly can not have $K_{3,3}$ as a minor. Similarly, it can be easily checked that none of $K^-_{4,4}$ and $K_{4,4}$ has K_5 as a minor. □

We are ready now to prove the main theorem of this section.

Theorem 4.4. *A cographic matroid M satisfies the following conditions:*

(i) it is 3*-connected,*

(ii) it has a minor isomorphic to $M^*(G_{17})$ or $M^*(G_{19})$, and

(iii) for any $Y \in \mathscr{C}^*(M)$, $M \backslash Y$ is graphic

if and only if $M \cong M^*(G)$, where $G \in \{K_{3,n}, K_{3,n}^{+1}, K_{3,n}^{+2}, K_{3,n}^{+3}, K_{4,4}^{-}, K_{4,4}\}$ with $n \geq 5$.

Proof: By Theorems 4.2 and 4.3, we conclude that the solution to the problem P_2 are the graphs in $\mathscr{G} = \{K_{3,n}, K_{3,n}^{+1}, K_{3,n}^{+2}, K_{3,n}^{+3}, K_{4,4}^{-}, K_{4,4}\}$ where $n \geq 5$. Since the problem P_0 is equivalent to P_2 we have that the "if" part follows.

For the "only if" part, clearly the cographic matroids of the graphs in $\mathscr{G}$ defined in problem P_0 are those satisfying the conditions of the theorem. We have shown that P_0 is equivalent to P_2. Thus, by Theorems 4.2 and 4.3, we have that if M satisfies conditions (i), (ii) and (iii) then M is isomorphic to the cographic matroid associated with a graph in $\mathscr{L} \cup \mathscr{M} = \{K_{3,n}, K_{3,n}^{+1}, K_{3,n}^{+2}, K_{3,n}^{+3}, K_{4,4}^{-}, K_{4,4}\}$. $\square$

We are now ready to present an algorithm which given a cographic matroid M with graphic cocircuits determines whether M is signed-graphic or not.

RECOGNITION ALGORITHM

Input: A cographic matroid M with graphic cocircuits.
Output: The matroid M is identified as signed-graphic or not.

Step 1. Decompose M into 3-connected cographic minors $M_1, \ldots, M_l$ $(l \geq 1)$of M via $1-$ and $2-$sums using the decomposition algorithm provided in [13, 15].

Step 2. For each M_i $(i = 1, \ldots, l)$, construct the unique up to isomorphism graph H_i such that $M_i = M(H_i)$. This can be done by applying the algorithm appearing in [16, 2].

Step 3. Test if there exists an H_i being isomorphic to one of the graphs in $\mathscr{G} = \{K_{3,n}, K_{3,n}^{+1}, K_{3,n}^{+2}, K_{3,n}^{+3}, K_{4,4}^{-}, K_{4,4}\}$. If yes, then M is not signed-graphic; otherwise, M is signed-graphic.

Let A be an $m \times n$ binary matrix such that $M \cong M[A]$ and let $w(A)$ be the number of nonzeros of A. Then, there exists an $O((m+n) \cdot w(A))$ time algorithm for step 1 (see [14]) and an $O(m \cdot w(A))$ time algoritm for step 2 (see [2]). Checking whether a graph is isomorphic to some graph in $\mathscr{G}$ is easy (i.e. it can be carried out in polynomial time) due to the special structure of the graphs in $\mathscr{G}$; specifically, it is trivial to check if a graph H_i given by step 2 is isomorphic with $K_{4,4}^{-}$ or $K_{4,4}$ while H_i must have n mutually non-adjacent vertices of degree 3 and a particular adjacency relation between the remaining 3 vertices in order to be isomorphic to a graph in $\{K_{3,n}, K_{3,n}^{+1}, K_{3,n}^{+2}, K_{3,n}^{+3}\}$. Regarding the storage of graphs and matrices, the simple data structures used for graphs and matrices in [14] are employed. The proof of correctness of this algorithm goes as follows. Since $M^*(G_{17})$ and $M^*(G_{19})$ are 3-connected matroids we have that if such a matroid was a minor of M then it must also be a minor of some matroid M_j in $\{M_1, \ldots, M_l\}$ (see [8, 15]). Therefore, M has an $M^*(G_{17})-$ or an $M^*(G_{19})-$minor if and only if some M_j in $\{M_1, \ldots, M_l\}$ has such a minor. Since, by Lemma 3.4, "having graphic cocircuits" is a minor-closed property we have that each of $M_1, \ldots, M_l$ has graphic cocircuits. Thus, by Theorem 4.4, M_j has a minor isomorphic to $M^*(G_{17})$ or $M^*(G_{19})$ if and only if M_j is isomorphic to some cographic matroid associated with a graph in $\{K_{3,n}, K_{3,n}^{+1}, K_{3,n}^{+2}, K_{3,n}^{+3}, K_{4,4}^{-}, K_{4,4}\}$ (where $n \geq 5$).

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